XXV Marató de Problemes FME

Problema 255

Solució estrictament escrita pels organitzadors i per res extreta d'internet

Let $f \neq 0$ be a polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \dots$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geq 0$. Prove that there exists a number N such that for every $n \geq N$, all roots of f_n are real.

Solution. For the proof, we need the following

Lemma 1. For any polynomial g, denote by $d(g)$ the minimum distance of any two of its real zeros ($d(g) = \infty$ if g has at most one real zero). Assume that g and $g + g'$ both are of degree $k \geq 2$ and have k distinct real zeros. Then $d(g + g') \geq d(g)$.

Proof of Lemma 1: Let $x_1 < x_2 < \cdots < x_k$ be the roots of g. Suppose a, b are roots of $g + g'$ satisfying $0 < b - a < d(g)$. Then, a, b cannot be roots of g, and

$$\frac{g'(a)}{g(a)} = \frac{g'(b)}{g(b)} = -1. \tag{1}$$

Since $\frac{g'}{g}$ is strictly decreasing between consecutive zeros of g, we must have $a < x_j < b$ for some j.

For all $i = 1, 2, \dots, k-1$ we have $x_{i+1} - x_i > b - a$, hence $a - x_i > b - x_{i+1}$. If $i < j$, both sides of this inequality are negative; if $i \geq j$, both sides are positive. In any case, $\frac{1}{a - x_i} < \frac{1}{b - x_{i+1}}$, and hence

$$\frac{g'(a)}{g(a)} = \sum_{i=1}^{k-1} \frac{1}{a - x_i} + \underbrace{\frac{1}{a - x_k}}_{<0} < \sum_{i=1}^{k-1} \frac{1}{b - x_{i+1}} + \underbrace{\frac{1}{b - x_1}}_{>0} = \frac{g'(b)}{g(b)}$$

This contradicts (1).

Now we turn to the proof of the stated problem. Denote by m the degree of f. We will prove by induction on m that f_n has m distinct real zeros for sufficiently large n. The cases $m = 0, 1$ are trivial; so we assume $m \geq 2$. Without loss of generality we can assume that f is monic. By induction, the result holds for f', and by ignoring the first few terms we can assume that f_n' has $m - 1$ distinct real zeros for all n. Let us denote these zeros by $x_1^{(n)} > x_2^{(n)} > \cdots > x_{m-1}^{(n)}$. Then f_n has minima in $x_1^{(n)}, x_3^{(n)}, x_5^{(n)}, \dots$, and maxima in $x_2^{(n)}, x_4^{(n)}, x_6^{(n)}, \dots$. Note that in the interval $(x_{i+1}^{(n)}, x_i^{(n)})$, the function $f_{n+1}' = f_n' + f_n''$ must have a zero (this follows by applying Rolle's theorem to the function $e^x f_n'(x)$); the same is true for the interval $(-\infty, x_{m-1}^{(n)})$. Hence, in each of these $m - 1$ intervals, f_{n+1}' has *exactly* one zero. This shows that

$$x_1^{(n)} > x_1^{(n+1)} > x_2^{(n)} > x_2^{(n+1)} > x_3^{(n)} > x_3^{(n+1)} > \dots \tag{2}$$

Lemma 2. We have $\lim_{n\to\infty} f_n(x_j^{(n)}) = -\infty$ if j is odd, and $\lim\limits_{n\to\infty} f_n(x_j^{(n)}) = +\infty$ if j is even.

Lemma 2 immediately implies the result: For sufficiently large n, the values of all maxima of f_n are positive, and the values of all minima of f_n are negative; this implies that f_n has m distinct zeros.

Proof of Lemma 2: Let $d = \min\{d(f'), 1\}$; then by Lemma 1, $d(f'_n) \geq d$ for all n. Define $\varepsilon = \dfrac{(m-1)d^{m-1}}{m^{m-1}}$; we will show that

$$f_{n+1}(x_j^{(n+1)}) \geq f_n(x_j^{(n)}) + \varepsilon \quad \text{for } j \text{ even.} \tag{3}$$

(The corresponding result for odd j can be shown similarly.) Do to so, write $f = f_n$, $b = x_j^{(n)}$, and choose a satisfying $d \leq b - a \leq 1$ such that f' has no zero inside (a, b). Define ξ by the relation $b - \xi = \dfrac{1}{m}(b-a)$; then $\xi \in (a, b)$. We show that $f(\xi) + f'(\xi) \geq f(b) + \varepsilon$.

Notice, that

$$\begin{aligned}\frac{f''(\xi)}{f'(\xi)} &= \sum_{i=1}^{m-1} \frac{1}{\xi - x_i^{(n)}} \\ &= \sum_{i<j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{< \frac{1}{\xi - a}} + \frac{1}{\xi - b} + \sum_{i>j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{<0} \\ &< (m-1)\frac{1}{\xi - a} + \frac{1}{\xi - b} = 0.\end{aligned}$$

The last equality holds by definition of ξ. Since f' is positive and $\dfrac{f''}{f'}$ is decreasing in (a, b), we have that f'' is negative on (ξ, b). Therefore,

$$f(b) - f(\xi) = \int_\xi^b f'(t)dt \leq \int_\xi^b f'(\xi)dt = (b - \xi)f'(\xi)$$

Hence,

$$\begin{aligned}f(\xi) + f'(\xi) &\geq f(b) - (b-\xi)f'(\xi) + f'(\xi) \\ &= f(b) + (1 - (\xi - b))f'(\xi) \\ &= f(b) + (1 - \tfrac{1}{m}(b-a))f'(\xi) \\ &\geq f(b) + (1 - \tfrac{1}{m})f'(\xi).\end{aligned}$$

Together with

$$f'(\xi) = |f'(\xi)| = m \prod_{i=1}^{m-1} \underbrace{|\xi - x_i^{(n)}|}_{\geq |\xi - b|} \geq m|\xi - b|^{m-1} \geq \frac{d^{m-1}}{m^{m-2}}$$

we get

$$f(\xi) + f'(\xi) \geq f(b) + \varepsilon.$$

Together with (2) this shows (3). This finishes the proof of Lemma 2.