Problema 100:

Resigullet [illegible]

Obs: Identitats de φ

$$\varphi^2 = 1+\varphi$$

$$\varphi^{-1} = \varphi - 1$$

Obs: La beta function:

$$B(\alpha,\beta) = \int_0^\infty \frac{x^{\beta-1}}{(1+x)^{\alpha+\beta}}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\int_0^\infty \frac{dx}{(1+x^{\varphi})^{\varphi}} \underset{u=x^{\varphi}}{=} \frac{1}{\varphi}\int_0^\infty \frac{u^{\varphi-2}}{(1+u)^{\varphi}}\,du = B(1,\varphi-1) = \frac{\Gamma(1)\,\Gamma(\varphi-1)}{\Gamma(\varphi)}\cdot\frac{1}{\varphi}$$

$$u = x^{\varphi}$$

$$du = \varphi x^{\varphi-1}\,dx \Leftrightarrow dx = \frac{1}{\varphi}\frac{x}{x^{\varphi}}\,du = \frac{1}{\varphi}\frac{u^{1/\varphi}}{u}\,du = \frac{1}{\varphi}u^{\varphi-2}\,du$$

$\rightarrow$ Fent servir que $\Gamma(z)\cdot z = \Gamma(z+1)$ tenim: $(\Gamma(1) = 1)$

$$\frac{\overset{1}{\cancel{\Gamma(1)}}\,\Gamma(\varphi-1)}{\varphi\,\Gamma(\varphi)} = \frac{\Gamma(\varphi-1)\cdot(\varphi-1)}{\Gamma(\varphi)} = \frac{\Gamma(\varphi)}{\Gamma(\varphi)} = \boxed{1}$$

La clau està en fer el canvi de variable $u = \frac{1}{x}$.

$\Rightarrow du = -u^2 dx$

$$I := \int_0^\infty \frac{1}{(1+x^\varphi)^\varphi}\, dx = \int_\infty^0 \frac{1}{\left(1+\frac{1}{u^\varphi}\right)^\varphi} \cdot \frac{-du}{u^2} =$$

$$= \int_0^\infty \frac{1}{(u^\varphi+1)^\varphi} \cdot \frac{1}{\left(\frac{1}{u^\varphi}\right)^\varphi} \cdot u^{-2}\, du = \int_0^\infty \frac{1}{(1+u^\varphi)^\varphi} \cdot u^{\varphi^2-2}\, du$$

I ara aprofitem que $\varphi^2 = \varphi + 1 \Rightarrow \varphi^2 - 2 = \varphi - 1$.

Per tant,

$$I = \int_0^\infty \frac{1}{(1+u^\varphi)^\varphi}\, u^{\varphi-1}\, du = \left[\frac{(1+u^\varphi)^{-\varphi+1}}{-\varphi+1} \cdot \frac{1}{\varphi} \right]_0^\infty =$$

$$= \frac{1}{\varphi(-\varphi+1)} \left[0 - 1 \right] = \frac{1}{\varphi(\varphi-1)} = \frac{1}{\varphi^2 - \varphi} = \frac{1}{1} = 1 \quad \blacksquare$$