~~12~~ 66. EULERDOS.

- $\xi_6 = e^{2\pi i/6}$

$$(\xi_6^k)^0 + (\xi_6^k)^1 + \dots + (\xi_6^k)^5 = \begin{cases} 6 & \text{if } k \text{ is divisible by } 6 \\ \dfrac{(\xi_6^k)^6 - 1}{\xi_6^k - 1} = 0 & \text{else.} \end{cases}$$

- $\xi_{12} = e^{2\pi i/12}$

$$(\xi_{12}^k)^0 + (\xi_{12}^k)^1 + \dots + (\xi_{12}^k)^{11} = \begin{cases} 12 & \text{if } k \text{ is divisible by } 12 \\ \dfrac{(\xi_{12}^k)^{12} - 1}{\xi_{12}^k - 1} = 0 & \text{else} \end{cases}$$

Then, notice $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$, then

$$S_6 = e^{\xi_6^0} + e^{\xi_6^1} + \dots + e^{\xi_6^5} = \sum_{k=0}^{5} \sum_{n\geq 0} \frac{(\xi_6^k)^n}{n!} = \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{5} (\xi_6^k)^n$$

where we used absolute convergence of S_6 to swap summations

$$\Rightarrow S_6 = \sum_{\substack{n\geq 0 \\ 6\mid n}} \frac{1}{n!} 6 = 6 \cdot \sum_{n\geq 0} \frac{1}{(6n)!}$$

Also, similarly

$$S_{12} = e^{\xi_{12}^0} + e^{\xi_{12}^1} + \dots + e^{\xi_{12}^{11}} = \sum_{k=0}^{11} \sum_{n\geq 0} \frac{(\xi_{12}^k)^n}{n!} = \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{11} (\xi_{12}^k)^n$$

$$\Rightarrow S_{12} = \sum_{\substack{n\geq 0 \\ 12\mid n}} \frac{12}{n!} = 12 \sum_{n\geq 0} \frac{1}{(12n)!}$$

Thus,

$$S = \sum_{n\geq 0} \frac{(-1)^n}{(6n)!} \overset{\text{abs. conv}}{=} -\sum_{n\geq 0} \frac{1}{(6n)!} + \sum_{n\geq 0} \frac{1}{(12n)!} \cdot 2 = \frac{S_6}{6} - \frac{2 S_{12}}{12}$$

$$\Rightarrow S = \frac{1}{6}\left(-(e^{\xi_6^0} + e^{\xi_6^1} + \dots + e^{\xi_6^5}) + (e^{\xi_{12}^0} + e^{\xi_{12}^1} + \dots + e^{\xi_{12}^{11}})\right)$$

Notice $\xi_{12}^{2k} = \xi_6^k$, so

$$S = \frac{1}{6}\left(e^{\xi_{12}^1} + e^{\xi_{12}^3} + \dots + e^{\xi_{12}^{11}}\right)$$

$$= \frac{1}{6}\left(e^{e^{\pi i/12}} + e^{e^{\pi i/4}} + e^{e^{5\pi i/12}} + e^{e^{7\pi i/12}} + e^{e^{3\pi i/4}} + e^{e^{11\pi i/12}}\right)$$

$$= \frac{1}{3} e^{-\sqrt{3}/2} \cos\left(\tfrac{1}{2}\right) + \frac{1}{3} e^{\sqrt{3}/2} \cos\left(\tfrac{1}{2}\right) + \frac{1}{3}\cos(1).$$

$$\simeq 0.9986111\ldots$$