5/19/2023 A triple Integral

$$I = \int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{e^x+e^y+e^z-1} = \frac{13}{4}\zeta(3)$$

Proof

$$e^x = u,\quad e^y = v,\quad e^z = w$$

$$I = \int_1^\infty\int_1^\infty\int_1^\infty \frac{1}{u+v+w-1}\,\frac{du\,dv\,dw}{uvw}, \qquad \frac{1}{u+v+w-1} = \int_0^\infty e^{-t(u+v+w-1)}\,dt$$

$$= \int_0^\infty dt\, e^{t}\left[\int_1^\infty \frac{e^{-tu}}{u}\,du\right]^3$$

$$= \int_0^\infty dt\, e^{+t}\, E_1^3(t)$$

where $E_1(t) \equiv \int_1^\infty \frac{e^{-xt}}{x}\,dx = \Gamma(0,t) = \int_t^\infty \frac{e^{-u}}{u}\,du$

$$E_1(t) = -\gamma - \ln t - \sum_{n=1}^\infty \frac{(-1)^n t^n}{n!\,n}, \qquad E_1(t) = \frac{e^{-t}}{t}\left[1 - \frac{1}{t} + \frac{2}{t^2} + \dots\right.$$

$$\lim_{t\to 0} t E_1(t) = 0, \qquad \lim_{t\to\infty} E_1(t) e^t = 0$$

$$E_1(t) \equiv -\int_\infty^t \frac{e^{-u}}{u}\,du, \qquad \frac{dE_1(t)}{dt} = -\frac{e^{-t}}{t}.$$

$$I = \int_0^\infty dt\, e^t E_1^3(t) = \int_0^\infty dt\, E_1^3(t)\, d\left[e^t - 1\right]$$

$$= E_1^3(t)\left[e^t-1\right]\Big|_0^\infty - \int_0^\infty dt\left[3E_1^2(t)\cdot\frac{dE_1(t)}{dt}\left(e^t-1\right)\right.$$

$$= 3\int_0^\infty \frac{e^t-1}{t}\, e^{-t} E_1^2(t)\,dt$$

$$= 3\int_0^\infty \frac{1-e^{-t}}{t}\, E_1^2(t)\,dt$$

$$= -3\int_0^\infty \frac{1}{t}\sum_{n=1}^\infty \frac{(-1)^n t^n}{n!}\, E_1^2(t)\,dt = -3\sum_{n=1}^\infty \frac{(-1)^n}{n!\,n}\int_0^\infty t^{n-1} E_1^2(t)\,dt$$

$$= -3\sum_{n=1}^\infty \frac{(-1)^n}{n!\,n}\left\{2\int_0^\infty E_1(t)\frac{e^{-t}}{t}\, t^n\,dt\right\} \qquad \text{(IBP)}$$

$$= -6\sum_{n=1}^\infty \frac{(-1)^n}{n!\,n}\int_0^\infty E_1(t)\, t^{n-1} e^{-t}\,dt$$

$$\int_0^\infty E_1(t)\, t^{n-1} e^{-t}\,dt = \int_0^\infty \frac{e^{-t}}{t}\, t^n\,dt \int_1^\infty \frac{e^{-xt}}{x}\,dx = \int_0^\infty \frac{e^{-t}}{t}\, t^n\,dt \int_0^\infty \frac{e^{-(y+1)t}}{y+1}\,dy$$

$$(x \to y+1)$$

$$= \int_0^\infty e^{-2t} t^{n-1} \, dt \int_0^\infty \frac{e^{-yt}}{y+1} \, dy = \int_0^\infty \frac{dy}{y+1} \int_0^\infty t^{n-1} e^{-(y+2)t} \, dt$$

$$= \int_0^\infty \frac{dy}{y+1} \frac{\Gamma(n)}{(y+2)^n} = (n-1)! \int_0^\infty dy \frac{1}{(y+1)(y+2)^n}$$

$$\Rightarrow I = -6 \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \int_0^\infty \frac{dy}{(y+1)(y+2)^n} = -6 \int_0^\infty dy \frac{1}{y+1} \mathrm{Li}_2\left(-\frac{1}{y+2}\right)$$

Let $\frac{1}{y+2} = t$, or $y+2 = \frac{1}{t}$

$$I = -6 \int_{\frac{1}{2}}^{0} \frac{\mathrm{Li}_2(-t)}{\left(\frac{1}{t}-1\right)} \left(-\frac{1}{t^2}\right) dt = -6 \int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{(1-t)t} \, dt$$

We now need to show

$$-6 \int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{(1-t)t} \, dt = \frac{13}{4} \zeta(3), \quad \text{or} \quad \underline{\int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{(1-t)t} = -\frac{13}{24} \zeta(3).}$$

$$\int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{t(1-t)} \, dt = \int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{t} \, dt + \int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{1-t} \, dt$$

$$= \mathrm{Li}_3\left(-\tfrac{1}{2}\right) - \mathrm{Li}_2\left(-\tfrac{1}{2}\right) \ln\left(\tfrac{1}{2}\right) - \int_0^{\frac{1}{2}} \frac{\ln(1-t)\ln(1+t)}{t} \, dt$$

$$= \mathrm{Li}_3\left(-\tfrac{1}{2}\right) + \ln(2) \mathrm{Li}_2\left(-\tfrac{1}{2}\right) - J$$

Where $J = \int_0^{\frac{1}{2}} \frac{\ln(1-t)\ln(1+t)}{t} \, dt$

$$= \frac{1}{2} \int_0^{\frac{1}{2}} \frac{\ln^2(1-x^2)}{x} \, dx - \frac{1}{2} \int_0^{\frac{1}{2}} \frac{\ln^2(1-x)}{x} \, dx - \frac{1}{2} \int_0^{\frac{1}{2}} \frac{\ln^2(1+x)}{x} \, dx$$

$$= \frac{1}{2} \Big\{ -\ln 2 \ln^2 \tfrac{3}{4} + \ln^3 2 + 2\ln 2 \, \mathrm{Li}_2\left(\tfrac{1}{2}\right) + \ln\tfrac{3}{4} \mathrm{Li}_2\left(\tfrac{3}{4}\right) \cancel{+ 2\mathrm{Li}_3\left(\tfrac{3}{4}\right)} - \mathrm{Li}_3\left(\tfrac{3}{4}\right) - 3\zeta(3)$$

$$+ 2\mathrm{Li}_3\left(\tfrac{1}{2}\right) + \ln 2 \ln^2\left(\tfrac{3}{2}\right) + \tfrac{2}{3} \ln^3\left(\tfrac{3}{2}\right) + 2\ln\tfrac{3}{2} \mathrm{Li}_2\left(\tfrac{2}{3}\right) + 2\mathrm{Li}_3\left(\tfrac{2}{3}\right) \Big\}$$

Using polyLog functional identities,

$$2\mathrm{Li}_2\left(\tfrac{1}{3}\right) - \mathrm{Li}\left(-\tfrac{1}{3}\right) = \frac{\pi^2}{6} - \frac{1}{2}\ln^2 3$$

$$\underline{2\mathrm{Li}_3\left(\tfrac{1}{3}\right) - \mathrm{Li}_3\left(-\tfrac{1}{3}\right) = \frac{13}{6}\zeta(3) - \frac{\pi^2}{6}\ln 3 + \frac{1}{6}\ln^3 3}$$

After some massive cancellations, we have

$$\int_0^{\frac{1}{2}} \frac{\mathrm{Li}_2(-t)}{t(1-t)} \, dt = -\frac{13}{24} \zeta(3) \qquad \text{QED!}$$