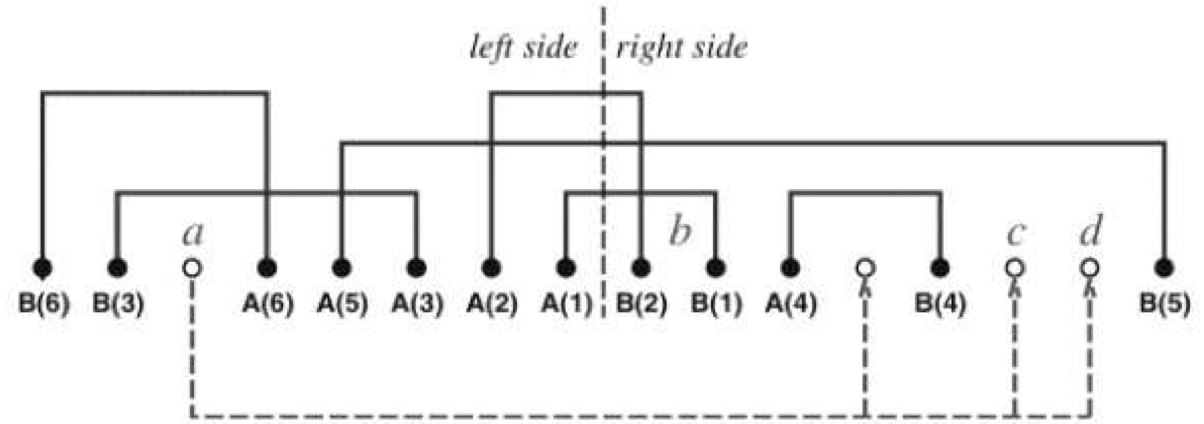
The combinatorial proof below was found by Joyce Justicz, then taking a graduate reading course with me at Emory University. Suppose the interval endpoints are chosen from $\{1, 2, \ldots, 2n\}$. We will label the points $A(1)$, $B(1)$, $A(2)$, $B(2)$, …, $A(n-2)$, $B(n-2)$ recursively as follows. Referring to points $\{n+1, \ldots, 2n\}$ as the *right side* and $\{1, \ldots, n\}$ as the *left side*, we begin by setting $A(1) = n$ and letting $B(1)$ be its mate. Suppose we have assigned labels up to $A(j)$ and $B(j)$, where $B(j)$ is on the left side; then $A(j+1)$ is taken as the left-most point on the right side not yet labeled, and $B(j+1)$ as its mate. If $B(j)$ is on the right side, $A(j+1)$ is the right-most unlabeled point on the left side and again $B(j+1)$ is its mate.

If $A(j) < B(j)$, we say that the jth interval "went right," otherwise it "went left." Points labeled $A(\cdot)$ are said to be *inner* endpoints, the others *outer*.



It is easily checked by induction that after the labels $A(j)$ and $B(j)$ have been assigned, either an equal number of points have been labeled on each side (in case $A(j) < B(j)$) or two more points have been labeled on the left (in case $A(j) > B(j)$).

When the labels $A(n-2)$ and $B(n-2)$ have been assigned, four unlabeled endpoints remain, say $a < b < c < d$. Of the three equiprobable ways of pairing them up, we claim two of them result in a "big" interval which intersects all others, and the third does not.

In case $A(n-2) < B(n-2)$, we have a and b on the left and c and d on the right, else only a is on the left. In either case, all inner endpoints lie between a and c, else one of them would have been labeled. It follows that the interval $[a, c]$ meets all others, and likewise $[a, d]$, so unless a is paired with b, we get a big interval.

Suppose on the other hand that the pairing is indeed $[a, b]$ and $[c, d]$. Neither of these can qualify as a big interval since they do not intersect each other; suppose some other interval qualifies, say $[e, f]$, labeled by $A(j)$ and $B(j)$.

When a and b are on the left, the inner endpoint $A(j)$ lies between b and c, thus $[e, f]$ cannot intersect both $[a, b]$ and $[c, d]$, contradicting our assumption.

In the opposite case, since $[e, f]$ meets $[c, d]$, f is an outer endpoint (so $f = B(j)$) and $[e, f]$ went right; since the last labeled pair went left, there is some $k > j$ for which $[A(k), B(k)]$ went left, but $[A(k-1), B(k-1)]$ went right. Then $A(k) < n$, but $A(k) < A(j)$ since $A(k)$ is a later-labeled, left-side inner point. But then $[A(j), B(j)]$ does not, after all, intersect $[B(k), A(k)]$, and this final contradiction proves the result. ♡

With slightly more care one can use this argument to show that for $k < n$, the probability that in a family of n random intervals there are at least k which intersect all others is

$$\frac{2^k}{\binom{2k+1}{k}}$$

independent, again, of n. The "binomial coefficient" $\binom{n}{k}$ stands for the number of subsets of size k from a set of size n, and is equal to $n(n-1)(n-2)\cdots(n-k+1)/k((k-1)(k-2)\cdots 1$.