1646. Autistralia Titus

Sigui $\{a_n\}$ Eloi i $\{b_n\}$ Bernat.

Definim $\{\alpha_n\}$ i $\{\beta_n\}$ tal que

$$a_n = \alpha_1\alpha_2\cdots\alpha_n \quad \left(\text{I.e.,} \quad \alpha_n := \frac{a_n}{a_{n-1}}\right).$$
$$b_n = \beta_1\beta_2\cdots\beta_n. \quad \text{amb} \quad \underline{\alpha_1=1, \alpha_2=2, \beta_1=1, \beta_2=3}.$$

~~Això~~ Això està ben definit ja que $a_i, b_i \neq 0$ ja que són creixents.

<u>Claim $\beta_k = 2k-1$.</u>

<u>Pf</u>. Trivial per inducció:

$$b_n = 2b_{n-1} + (2n-3)^2 b_{n-2}$$

$$\underset{\text{I.H}}{=} 2(1\cdot3\cdot\ldots\cdot(2n-3)) + (2n-3)^2(1\cdot3\cdot\ldots\cdot(2n-5))$$

$$= (1\cdot3\cdot\ldots\cdot(2n-3))(2+2n-3) = 1\cdot3\cdot\ldots\cdot(2n-3)(2n-1)$$

$$\Rightarrow \beta_n = 2n-1. \quad \blacksquare$$

Ara definim $p_n = \dfrac{\alpha_n}{2n-1} \Rightarrow \alpha_n = p_n(2n-1)$. Per tant tenim

$$\frac{a_n}{b_n} = \frac{\alpha_1\cdots\alpha_n}{\beta_1\cdots\beta_n} = \frac{p_1\cancel{(2-1)}\cdots p_n\cancel{(2n-1)}}{\cancel{(2-1)}\cdots\cancel{(2n-1)}} = p_1p_2\cdots p_n.$$

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Per tant $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \prod_{i=1}^{n} p_i$.

~~Nota que~~ Busquem la recurrència dels $\{p_i\}$. Tenim

$$a_n = 2a_{n-1} + (2n-3)^2 a_{n-2}$$

$$\Updownarrow$$

$$\frac{a_n}{a_{n-1}} = 2 + (2n-3)^2 \frac{a_{n-2}}{a_{n-1}}$$

$$\Updownarrow$$

$$\alpha_n = 2 + (2n-3)^2 \frac{1}{\alpha_{n-1}}$$

$$\Updownarrow$$

$$p_n(2n-1) = 2 + (2n-\cancel{3})^{\cancel{2}} \frac{1}{p_{n-1}\cancel{(2n-3)}}$$

$$\Updownarrow$$

$$p_n = \frac{2}{2n-1} + \frac{2n-3}{2n-1}\frac{1}{p_{n-1}}, \quad \text{amb } p_1 = 1,\ p_2 = \frac{2}{3}.$$

Si $q_n := p_1 \cdots p_n$, llavors (Nota $q_1 = 1$, $q_2 = \frac{2}{3}$).

$$q_n = q_{n-1} p_n = q_{n-1}\left(\frac{2}{2n-1} + \frac{2n-3}{2n-1}\frac{1}{p_{n-1}}\right) =$$

$$= q_{n-1}\frac{2}{2n-1} + \frac{2n-3}{2n-1} q_{n-2}$$

$$\Rightarrow \quad q_n - q_{n-1} = -\frac{2n-3}{2n-1}(q_{n-1} - q_{n-2})$$

$$= \frac{\cancel{2n-3}}{2n-1}\cdot\frac{2n-5}{\cancel{2n-3}}(q_{n-2} - q_{n-3}) =$$

$$= \dots = (-1)^{n+1}\frac{\cancel{2n-3}}{2n-1}\cdot\frac{\cancel{2n-5}}{\cancel{2n-3}}\cdot \cancel{\dots} \cdot \frac{3}{\cancel{5}}(q_2 - q_1)$$

$$= (-1)^{n+1}\frac{3}{2n-1}\cdot\frac{-1}{3} = (-1)^n \frac{1}{2n-1}.$$

Per tant

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} (q_n - q_{n-1}) + (q_{n-1} - q_{n-2}) + \dots + (q_2 - q_1) + q_1$$

$$= \cancel{\lim} \ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

$$= \frac{\pi}{4}$$

arctan(1), very well-known.