P8.

Sol 1. Let me to prove the result with the following well-known identity:

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \binom{n-4}{3} + \dots + \binom{n-\lfloor\frac{n}{2}\rfloor-1}{\lfloor\frac{n}{2}\rfloor}$$

This is obviously well known.

We also use the following well know inequality

$$\binom{k}{m} \le \binom{k+1}{m} \le \binom{k+2}{m} \le \dots \quad \forall k \ge m \ge 0.$$

Now, let me to prove what we need.

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \dots + \binom{n-\lfloor\frac{n}{2}\rfloor-1}{\lfloor\frac{n}{2}\rfloor}$$

$$\le \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{\lfloor\frac{n}{2}\rfloor}$$

$$\le \sum_{k=0}^{n} \binom{n-1}{k} \le 2^{n-1} \le 2^n$$

Sol 2. We use the following very well-known identity of Cesàro:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3k}$$

If you don't know it skill issue.
By induction; if $F_0, \ldots, F_{3n-1}$ all satisfy with $n \geq 2$, we see $F_{3n}, F_{3n+1}, F_{3n+2}$ also.

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k \leq \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \cdot 2^k = 5^n$$

$$\Rightarrow F_{3n} \leq 5^n \leq 2^{3n} \quad \checkmark$$

We need F_{3n+1} and F_{3n+2}.
But clerly, we have, as before (and since $n+1 \leq 3n$)

$$F_{3n+3} = \sum_{k=0}^{n+1} \binom{n+1}{k} 2^k F_k \leq (4+1)^{n+1} = 5^{n+1}$$

and so

$$F_{3n+1} \leq F_{3n+3} \leq 5^{n+1} \leq 2^{3n+1} \quad \checkmark$$

$$F_{3n+2} \leq F_{3n+3} \leq 5^{n+1} \leq 2^{3n+2} \quad \checkmark$$

So $F_{3n}, F_{3n+1}, F_{3n+2}$ satisfy the result. | Base cases $F_0, F_1, \ldots, F_5$ trivial. | True if $n \geq 2$ by easy induction