P47.

Siguin $X_i \sim U(0,1)$ i.i.d. i $k>0$.

Definim $N = \min \{ n : k - X_1 - \cdots - X_n < 0 \}$.

És a dir, $N(\omega) = n \Leftrightarrow k - X_1 - \cdots - X_n < 0$ i $k - X_1 - \cdots - X_{n-1} \geq 0$.

Definim $f(k) = \mathbb{E}[N]$ segons el valor de k.

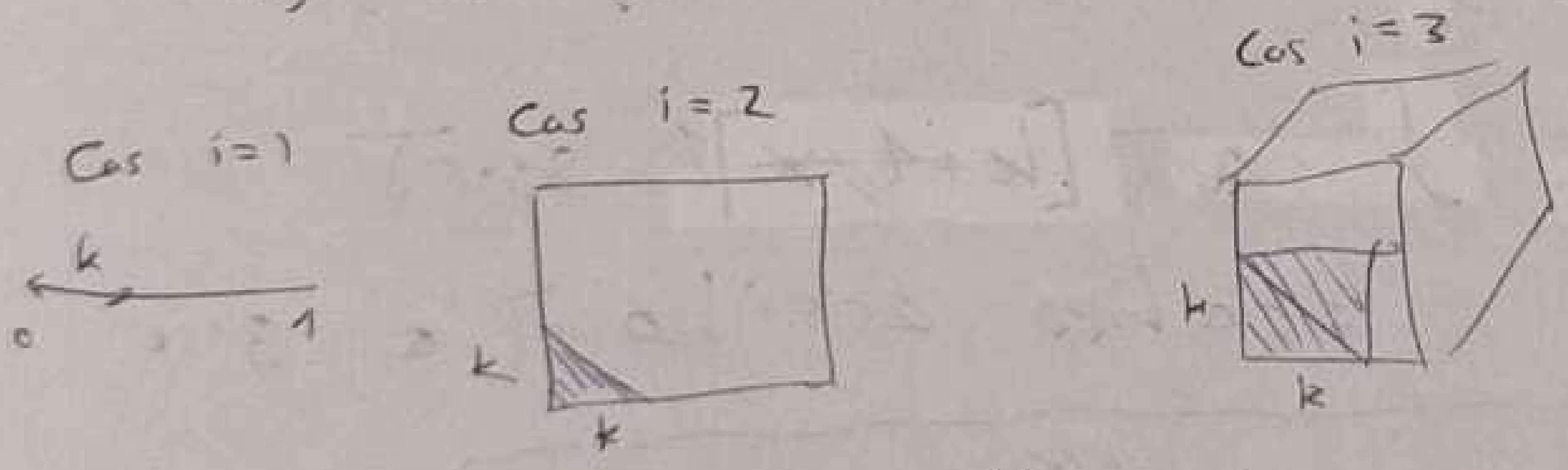
Claim: $f(k) = e^k$ per $k \in (0,1]$.

D/ Escrivim $N = \sum_{i \geq 0} Z_i$, on $Z_i = \begin{cases} 1 & \text{si } k - X_1 - \cdots - X_i \geq 0 \\ 0 & \text{altrament} \end{cases}$

Aleshores, $\mathbb{E}[N] = \sum_{i \geq 0} \mathbb{E}[Z_i]$

Ara, $\mathbb{E}[Z_i] = P(X_1 + \cdots + X_i \leq k) = \int_{x_1 + \cdots + x_i \leq k} dx_1 \cdots dx_i$ =

Aquesta integral és igual al volum d'un i-símplex amb vèrtexs $0, k\vec{e_1}, k\vec{e_2}, \ldots, k\vec{e_i}$, el qual és $\frac{k^i}{i!}$.



Per tant $\mathbb{E}[N] = \sum_{i \geq 0} \frac{k^i}{i!} = e^k$

Trobarem $f(x)$ a trossos: ~~Da~~

Claimejem que té forma

$$f(x)=P_k(x-k)e^{x-k}, \quad x\in[k,k+1]$$

per $k\in\mathbb{N}$ i $P_k\in\mathbb{R}[x]$. Eg. $P_0(x)=1$.

<u>Obs.</u> Tenim la equació, si $x\geq 0$

$$f(x+1)=1+\int_x^{x+1} f(t)\,dt$$

$$\Rightarrow \boxed{f'(x)=f(x)-f(x-1) \quad \forall x\geq 1}$$

Si fixes $x\in[k+1,k+2]$, obtens: (factor integrant)

$$f(x)=\frac{1}{e^{-x}}\left(C_1+\int_{k+1}^{x} e^{-k}\left(-P_k(t-k-1)\right)dt\right)$$

on C_1 constant. Mirant $x=k+1$ surt $C_1=e^{-k}P_k(1)$, i acabes arribant a que

$$\boxed{P_{k+1}(x)=eP_k(1)-\int_0^x P_k(t)\,dt} \quad \forall x\in[0,1]$$

on $P_0(x)=1$. Això ens dona una recurrència pels P_k que ens permetrà calcular-los. Si voleu checkejar que no hem cagat els càlculs, ens surt $P_0=1$, $P_1=e-x$, $P_2=e(e-1)-xe+x^2/2$, ...

$P_k(x)$ té grau k. Escrivim $P_k(x) = \sum_{i=0}^{k} a_{k,i} x^i$.

Per $k \geq 1$, com $P_k'(x) = -P_{k-1}(x)$ tenim $i a_{k,i} = \begin{array}{l} -a_{k-1,i-1} \text{ per } i \geq 1 \\ \cancel{-a_{k,i+1} \text{ per } i \geq 1} \end{array}$

$\Rightarrow a_{k,i} = \frac{-1}{i} \cancel{a_{k,i+1}}\ a_{k-1,i-1}$

Aplicant aquesta identitat i cops, $a_{k,i} = \frac{(-1)^i}{i!} \cancel{a_{k,0}}\ a_{k-i,0}$

Sigui $c_0 = a_{k,0} =$ terme constant de $P_k(x)$.

Aleshores, $P_k(x) = \sum_{i=0}^{k} \frac{(-1)^i}{i!} c_{k-i} x^i$.

Per la fórmula recurrent $P_{k+1}(x) = eP_k(1) - \int_0^x P_k(t)\, dt$, el terme constant és $c_{k+1} = eP_k(1) = e\sum_{i=0}^{k} a_{k,i} = e\sum_{i=0}^{k} \frac{(-1)^i}{i!} c_{k-i} =$

$$= e \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k-i)!} c_i .$$

Definim $C(z) = \sum_{k\geq 0} c_k z^k$.

Aleshores, aplicant la recurrència trobada per c_k,

$\cancel{C(z) = c_0 + e \sum_{k\geq 0} \sum_{i=0}}$

$$C(z) = \sum_{k\geq 0} c_k z^k = c_0 + \sum_{k\geq 1} e \sum_{i=0}^{k-1} \frac{(-1)^{k-1-i}}{(k-1-i)!} c_i z^k =$$

$$= c_0 + e \sum_{i\geq 0} c_i \sum_{k\geq i+1} \frac{(-1)^{k-1-i}}{(k-1-i)!} z^{k-1-i} z^{i+1} \overset{m=k-1-i}{=}$$

$$= c_0 + e \underbrace{\sum_{i\geq 0} c_i z^{i+1}}_{z C(z)} \underbrace{\sum_{m\geq 0} \frac{(-1)^m}{m!} z^m}_{e^{-z}} = c_0 + z e^{1-z} C(z)$$

$$C(z) = 1 + z e^{1-z} C(z) \Rightarrow \qquad C(z) = \frac{1}{1 - z e^{1-z}}$$

$$C(z) = \sum_{a \geq 0} \left(z e^{1-z}\right)^a = \sum_{a \geq 0} z^a e^a e^{-az} = \sum_{a \geq 0} z^a e^a \sum_{b \geq 0} \frac{(-az)^b}{b!} =$$

$$= \sum_{a,b \geq 0} \frac{e^a (-a)^b}{b!} z^{a+b} = \sum_{k \geq 0} z^k \sum_{a=0}^{k} \frac{e^a (-a)^{k-a}}{(k-a)!}$$

Aleshores $\quad C_k = \sum_{a=0}^{k} \frac{e^a (-a)^{k-a}}{(k-a)!}$

$$P_k(x) = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k-i)!} x^{k-i} c_i = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k-i)!} x^{k-i} \sum_{a=0}^{i} \frac{e^a (-a)^{i-a}}{(i-a)!} =$$

$$= \sum_{a=0}^{k} e^a \sum_{i=a}^{k} \frac{(-1)^{k-i} x^{k-i} (-1)^{i-a} a^{i-a}}{(k-i)!\,(i-a)!} =$$

$$\underset{j=i-a}{=} \sum_{a=0}^{k} e^a (-1)^{k-a} \sum_{j=0}^{k-a} \frac{x^{k-a-j} a^j}{(k-a)!} \binom{k-a}{j} = \sum_{a=0}^{k} e^a (-1)^{k-a} \frac{1}{(k-a)!} (x+a)^{k-a} =$$

$$= \sum_{b=0}^{k} e^{k-b} \frac{(-1)^b}{b!} (x+k-b)^b$$

Per tant, per $x \in [k, k+1]$,

$$f(x) = P_k(x-k)\, e^{x-k} = \sum_{b=0}^{k} e^{x-b} \frac{(-1)^b}{b!} (x-b)^b$$

En resum, $\quad f(x) = \sum_{i \leq \lfloor x \rfloor} e^{x-i} \frac{(-1)^i}{i!} (x-i)^i$