P.108 Raticida. Three Men Orchestra

Considera el polinomi

$$p(x) := \sum_{\sigma \in \mathfrak{S}_n} \frac{\varepsilon(\sigma)}{\nu(\sigma)+1} x^{\nu(\sigma)+1}.$$

Busquem $p(1)$

Ara $$p'(x) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x^{\nu(\sigma)}$$

$$= \begin{vmatrix} x & 1 & \cdots & 1 \\ 1 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & x \end{vmatrix}$$

matriu $n \times n$.

Sigui $A = \begin{pmatrix} x & & 1 \\ & \ddots & \\ 1 & & x \end{pmatrix}$. $A-(x-1)Id = \begin{pmatrix} 1 \end{pmatrix}$ té rang 1, així que té VAP 0 de mult. $= n-1$. Per tant, A té VAP $x-1$ de mult. $n-1$.

A més, $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ és un VEP de VAP $x+n-1$.

~~Per tant~~

Per tant, el determinant és $(x-1)^{n-1}(x+n-1)$.

Integrem-ho:

$$\int (x-1)^{n-1}(x+n-1)\,dx = \frac{1}{n}(x-1)^n(x+n-1) - \int \frac{1}{n}(x-1)^n \cdot 1\,dx$$

$$= \frac{1}{n}(x-1)^n(x+n-1) - \frac{1}{n(n+1)}(x-1)^{n+1}.$$

Per tant,

$$p(x) = \left(\frac{1}{n}(x-1)^n(x+n-1) - \frac{1}{n(n+1)}(x-1)^{n+1}\right) + C.$$

Nota que $p(0)=0$, ja que $p(x)$ té termes $x^{\nu(\sigma)+1}$, $\nu(\sigma)\geq 0$, així que $\nu(\sigma)+1\geq 1$. Per tant,

$$0 = \left(\frac{1}{n}(-1)^n(n-1) - (-1)^{n+1}\frac{1}{n(n+1)}\right) + C$$

$$\Rightarrow C = (-1)^{n+1}\left(\frac{n-1}{n} + \frac{1}{n(n+1)}\right) = (-1)^{n+1}\frac{n^2-\cancel{1}+\cancel{1}}{n(n+1)}$$

$$= (-1)^{n+1}\frac{n}{n+1}.$$

Així,

$$\left[\sum_{\sigma\in\mathfrak{S}_n} \frac{\varepsilon(\sigma)}{\nu(\sigma)+1} = p(1) = C = (-1)^{n+1}\frac{n}{n+1}.\right]$$